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Weak Hopf C^*-algebras and depth two subfactors

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Abstract

We prove that for a finite index and depth two inclusion $N \subset M$ of II_1 factors, the relative commutants $N' \cap M_1$ and $M' \cap M_2$ admit mutually dual weak Hopf C^*-algebra structures. The proof is based on 'planar algebra techniques'. In the hyperfinite case, one can show that $N' \cap M_1$ acts on M with invariants N using this approach.

Keywords: Relative commutant; Planar algebra; Weak Hopf C^*-algebra

1. Introduction

Our goal in this paper is to give a pictorial proof of the existence of mutually dual weak Hopf C^*-algebra structures on $N' \cap M_1$ and $M' \cap M_2$ for a finite index and depth two inclusion $N \subset M$ of II_1 factors with basic construction tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$.

Results of this type have been obtained earlier and it might be useful to place this paper in context. It was asserted by Ocneanu [Cnn] and given a detailed proof by Szymański [Szy] that for an irreducible, finite index and depth two inclusion of II_1 factors, the relative commutants $A = N' \cap M_1$ and $B = M' \cap M_2$ have mutually dual Kac algebra ($= C^*$-Hopf algebra) structures. In attempting to generalise this result to the not necessarily irreducible case, Böhm et al. [BhmNllSzl] Böhm

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and Szlachányi [BhmSzl] isolated the fundamental notion of a weak Hopf C^*-algebra—henceforth abbreviated C^*-WHA—and proved many interesting results about these objects, some jointly with Wiesbrock [NllSzlWsb]. It was shown by Nikshych and Vainerman [NksVnr] that without assuming irreducibility, A and B have mutually dual C^*-WHA structures—however this was shown with a modified algebra structure on B. Moving away from the C^* setting, Kadison and Nikshych [KdsNks] showed that if $N \subset M$ is a certain kind of 'symmetric Markov extension' over an arbitrary field, then A and B have mutually dual WHA structures but the relationship between this duality and the $*$-structure—when such a structure is present—was not elucidated.

One of our motivations in writing this paper is to clarify this as a duality between C^*-WHAs. More important, however, is our approach, which is based on exploiting the marvellous structure discovered by Jones on the tower of relative commutants of an extremal finite index subfactor that is encapsulated in the notion of a planar algebra. Indeed, we believe that a comparison of our proof with existing proofs of related results should convince the reader of the power and beauty of planar algebras. The planar algebra approach has been successfully used in [DasKdy] to deal with the irreducible case.

We briefly summarise the contents of this paper. In Section 2 we recall the definition of a C^*-WHA and define what it means for a pair to be dual. A short summary of planar algebras and Jones' theorem forms Section 3. In Section 4, we prove our main theorem. In Section 5 we define the action.

2. Weak Hopf C^*-algebras

In this section, we define the notion of a C^*-WHA and show that this definition is equivalent to the one of [BhmNllSzl]. We also define duality of C^*-WHAs in a manner that is slightly different from that of [BhmNllSzl].

Definition 2.1. A C^*-WHA is a finite-dimensional complex vector space A, that has the structure of a C^*-algebra $(A, \mu, 1, *)$ and a (coassociative) coalgebra (A, Δ, ε) and is equipped with an endomorphism S such that the following axioms hold for all $a, b, c \in A$:

(1) Δ: $A \to A \otimes A$ is a (not necessarily unital) $*$-homomorphism;
(2) $(\Delta \otimes id)(\Delta(1)) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))$, i.e., $1_1 \otimes 1_2 \otimes 1_3 = 1_1 \otimes 1_2 1_{1'} \otimes 1_{2'}$;
(3) $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2 c)$;
(4) $\mu(S \otimes id)\Delta(a) = (id \otimes \varepsilon)((1 \otimes a)\Delta(1))$, i.e., $S(a_1)a_2 = 1_1 \varepsilon(a 1_2)$;
(5) S is an anti-algebra and anti-coalgebra map.

In this definition and for the rest of this paper, we will use Sweedler's notation for the comultiplication and its iterates but omit the summation sign. Thus, for instance, $\Delta(b) = b_1 \otimes b_2$.

Proposition 2.2. *Definition* 2.1 *of a* C^**-WHA is equivalent to that of Definition* 4.3 *of* [BhmNllSzl].

Proof. For a C^*-WHA in the sense of Definition 4.3 of [BhmNllSzl], conditions (1)–(4) of Definition 2.1 are a part of the axioms there while condition (5) follows from their Theorem 2.10. Conversely, given a C^*-WHA in the sense of Definition 2.1, conditions (2)–(4) along with their adjointed versions—i.e., taking $*$ and using (1)—show that all axioms of [BhmNllSzl] are satisfied with the possible exception of $Sa_1a_2Sa_3 = Sa$.

To prove this, we first recall the left and right counital maps defined by the equations: $\varepsilon_L(a) = \varepsilon(1_1 a)1_2$ and $\varepsilon_R(a) = 1_1\varepsilon(a1_2)$, and claim that $\varepsilon_R(a) = 1_1\varepsilon(1_2 S(a))$.

Given this claim, we see that

$$\begin{aligned} S(a_1)a_2S(a_3) &= \varepsilon_R(a_1)a_2 \\ &= 1_1\varepsilon(1_2S(a_1))S(a_2) \\ &= S(1_2)\varepsilon(S(1_1)S(a_1))S(a_2) \\ &= \varepsilon(S(a_1 1_1))S(a_2 1_2) \\ &= \varepsilon(S(a_1))S(a_2) \\ &= S(a), \end{aligned}$$

where the third equality holds since S is a anti-coalgebra map and the fourth holds since S is a anti-algebra map.

To prove the claim, note that (3) of Definition 2.1 implies that $\varepsilon(ab) = \varepsilon(\varepsilon_R(a)b)$ and $\varepsilon(ab) = \varepsilon(a\varepsilon_L(b))$, and therefore,

$$\begin{aligned} \varepsilon_R(a) &= 1_1\varepsilon(\varepsilon_R(a)1_2) \\ &= 1_1\varepsilon(1_2\varepsilon_R(a)) \\ &= 1_1\varepsilon(S(\varepsilon_R(a))S(1_2)) \\ &= 1_1\varepsilon(\varepsilon_L(S(a))\varepsilon_R(1_2)) \\ &= 1_1\varepsilon(\varepsilon_R(1_2)\varepsilon_L(S(a))) \\ &= 1_1\varepsilon(1_2S(a)), \end{aligned}$$

where the second equality holds because the images A_L and A_R of the maps ε_L and ε_R commute, the third because $\varepsilon = \varepsilon\circ S$ and the fourth because $S\circ\varepsilon_R = \varepsilon_L\circ S$ and $S = \varepsilon_R$ on A_L, both again being easy consequences of the weak multiplicativity of ε. This completes the proof. $\square$

Definition 2.3. A left (resp., right) integral in a C^*-WHA A is an element l (resp., r) $\in A$ such that $xl = \varepsilon_L(x)l$ (resp., $rx = r\varepsilon_R(x)$) $\forall x \in A$. An element h which is both a left and a right integral is called a 2-sided integral. Further, it is said to be normalised, if $\varepsilon_L(h) = 1 = \varepsilon_R(h)$. A normalised 2-sided integral is said to be a Haar integral.

A basic result of [BhmNllSzl] shows that in a C^*-WHA there is a unique Haar integral.

Definition 2.4. Two C^*-WHAs A and B are said to be dual to each other if there is a non-degenerate bilinear pairing $\langle \cdot | \cdot \rangle$: $A \times B \to \mathbb{C}$ such that the multiplication, comultiplication and antipode of A are dual, respectively, to the comultiplication, multiplication and antipode of B via the pairing and such that for $a \in A$, $b \in B$, $\langle a \,|\, b^* \rangle = \overline{\langle S(a^*) \,|\, b \rangle}$.

We should mention at this point that the usual $*$-structure defined on B via the pairing is given by $\langle a \,|\, b^* \rangle = \overline{\langle S(a)^* \,|\, b \rangle}$. This is related to that of the above definition by conjugation by a positive element, and hence the resulting C^* algebra structures are equivalent.

It should be clear—and is what we actually use—that given a non-degenerate pairing between two C^*-algebras A and B with the duals of multiplication and unit of B being defined to be the comultiplication and counit of A, if A is a C^*-WHA with antipode S and $\langle a \,|\, b^* \rangle = \overline{\langle S(a^*) \,|\, b \rangle}$, then A and B have the structures of a pair of dual C^*-WHAs.

3. Subfactors and planar algebras

This section is devoted to stating the fundamental theorem of Jones relating subfactors and planar algebras. For more details on the material of this section, the reader may consult [Jns,Lnd] or [KdyLndSnd].

The basic structure that underlies planar algebras is an action by the 'coloured operad of planar tangles' which concept we will now explain. Consider a set $Col = \{0_\pm, 1, 2, \dots\}$, elements of which we will call colours. We will not define a tangle but give several important examples and point out some features that they have. Figs. 1–9 are all examples of tangles.

Each tangle has an external box and some (possibly 0) ordered internal boxes. Normally, we will call the external box D_0 and the internal boxes $D_1, D_2, \dots$. Each box has an even number of points marked on its boundary (again possibly 0)—a box with $2k$ points on its boundary is called a k-box. If a box has at least one point marked on its boundary, one of the marked points is distinguished and marked with a $*$. The $*$ point of a box is considered as its first marked point and the rest are read off in clockwise order. There is also given a collection of curves each of which is either closed, or joins a marked point on one of the boxes to another such point. The

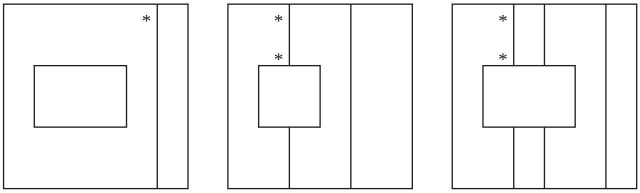


Fig. 1. Inclusion tangles: $I^1_{0_+}, I^2_1, I^3_2$.

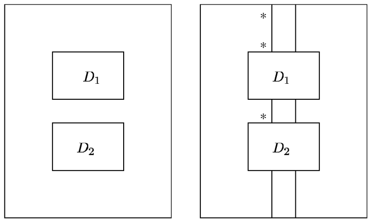


Fig. 2. Multiplication tangles: $M^{0_+}_{0_+,0_+}, M^2_{2,2}$.

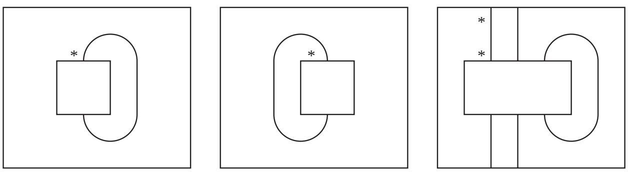


Fig. 3. Conditional expectation tangles: $E^{0_+}_1, E^{0_-}_1, E^2_3$.

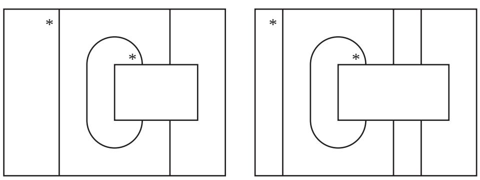


Fig. 4. Conditional expectation tangles: $(E')^2_2, (E')^3_3$.

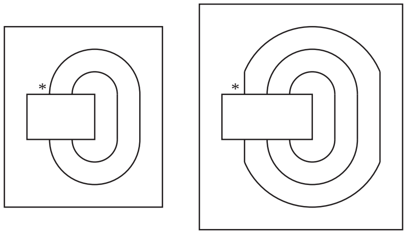


Fig. 5. Trace tangles: $tr^{0_+}_2, tr^{0_+}_3$.

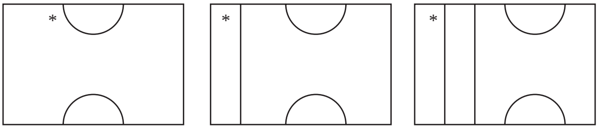


Fig. 6. Jones projection tangles: $\mathscr{E}^2, \mathscr{E}^3, \mathscr{E}^4$.

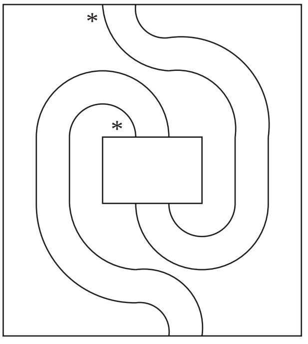


Fig. 7. The 2-rotation tangle: R_2^2.

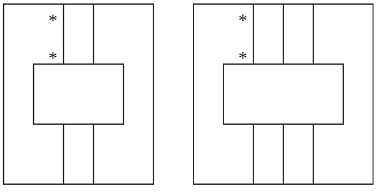


Fig. 8. The identity tangles: I_2^2, I_3^3.

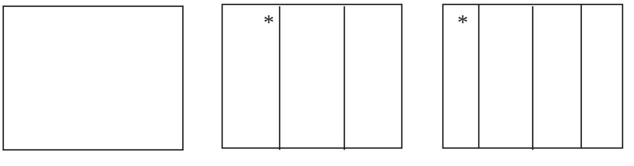


Fig. 9. The 1 tangles: $1^{0+}, 1^2, 1^3$.

whole picture is to be planar and each marked point on a box must be the end-point of one of the curves. Finally, there is given a black-and-white shading of the regions in such a way that moving away from (resp. towards) the $*$ on one of the internal boxes (resp. the external box) along the curve of which it is the end-point, the black region is to the right. A 0 box is said to be 0_+ box if the region touching its boundary is white and a 0_- box otherwise. A tangle is said to be a k-tangle if its external box is of colour k. As a matter of notation, tangles will be given names including subscripts and a superscript. The subscripts indicate the colours of the internal boxes and the superscript indicates the colour of the tangle. Given the subscripts and the superscript, the entire shading scheme is determined and will not be indicated. We will not distinguish between tangles that can be obtained from each other by a planar isotopy preserving the $*$'s, the shading and the ordering of the internal boxes.

The basic operation that one can perform on tangles is substitution of one into a box of another. So if T is a tangle that has an internal k box, say D_i, and S is a k-tangle, then we may substitute S in D_i to get a new tangle denoted $T\circ_{D_i}S$. The substitution is done in such a way that the '$*$'s match'. More generally, if T has internal boxes $D_{i_1}, \dots, D_{i_j}$ of colours $k_{i_1}, \dots, k_{i_j}$, respectively, and if $S_1, \dots, S_j$ are tangles of colours $k_{i_1}, \dots, k_{i_j}$, respectively, then we may form a tangle $T\circ_{(D_{i_1},\dots,D_{i_j})}(S_1, \dots, S_j)$ by the obvious substitutions. The collection of tangles along with the substitution operation is called the coloured operad of planar tangles.

A planar algebra P is an algebra over the coloured operad of planar tangles. By this we mean the following: P is a collection P_k of vector spaces for $k \in Col$ and maps Z_T: $P_{k_1} \otimes P_{k_2} \otimes \cdots \otimes P_{k_b} \to P_{k_0}$ for each k_0-tangle T with internal boxes of colours $k_1, k_2, \dots, k_b$. The collection of maps is to be 'compatible with substitution of tangles and renumbering of internal boxes' in an obvious manner. Further, planar algebras are required to be non-degenerate in the sense that for each $k \in Col$, the map $Z_{I_k^k}$ is the identity map of P_k.

A pleasant verification shows that given a planar algebra P, each P_k has the structure of an associative, unital algebra where the multiplication is given by $Z_{M_{k,k}^k}$ and the unit by $Z_{1^k}(1)$ and that the $Z_{I_k^{k+1}}$ are algebra homomorphisms from P_k to P_{k+1}.

Among planar algebras, the ones that we will be interested in are the subfactor planar algebras. For a detailed definition see [KdyLndSnd]. They are finite-dimensional and connected in the sense that each P_k is a finite-dimensional vector space and $P_{0_\pm}$ are one dimensional. They have a positive modulus δ in the sense that closed loops in a tangle T contribute a multiplicative factor of δ in Z_T. This implies that $Z_{I_k^{k+1}}$ are injective and we will identify P_k with $ran(Z_{I_k^{k+1}}) \subset P_{k+1}$. They are spherical in the sense that for a zero tangle T, the function Z_T is not just planar isotopy invariant but also an isotopy invariant of the tangle regarded as embedded on the surface of the 2-sphere. Further, each P_k is a C^*-algebra in such a way that for a k_0-tangle T with internal boxes of colours $k_1, k_2, \dots, k_b$ and $x_i \in P_{k_i}$

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*), \tag{3.1}$$

where T^* is the adjoint of the tangle T—which, by definition, is obtained from T by reflecting it, inverting the shading and changing the $*$ on all its k boxes to the reflected $2k$ position for each $k \geqslant 1$. Finally, the pictorial trace τ defined for $x \in P_k$ by

$$\tau(x).1^{0_+} = \delta^{-k} Z_{tr_k^{0_+}}(x)$$

is a faithful, positive, normalised trace.

We now state the theorem of Jones [Jns] relating subfactors and subfactor planar algebras.

Theorem 3.5. *Let*

$$N \subset M (= M_0) \subset^{e_1} M_1 \subset \cdots \subset^{e_k} M_k \subset^{e_{k+1}} \cdots$$

be the tower of the basic construction associated to an extremal subfactor with $[M:N] = \delta^2 < \infty$. *Then there exists a unique subfactor planar algebra* $P = P^{N \subset M}$ *of modulus* δ *satisfying the following conditions*:

(0) $P_k = N' \cap M_{k-1} \forall k \geqslant 1$-*where this is regarded as an equality of* *-*algebras which is consistent with the inclusions on the two sides*;
(1) $Z_{\mathscr{E}^{k+1}}(1) = \delta e_k \forall k \geqslant 1$;
(2) $Z_{(E')_k^k}(x) = \delta E_{M' \cap M_{k-1}}(x) \forall x \in N' \cap M_{k-1}, \forall k \geqslant 1$;
(3) $Z_{E_{k+1}^k}(x) = \delta E_{N' \cap M_{k-1}}(x) \forall x \in N' \cap M_k$; *and this is required to hold for all* k *in* Col, *where for* $k = 0_\pm$, *the equation is interpreted as*

$$Z_{E_1^{0_\pm}}(x) = \delta tr_M(x) \forall x \in N' \cap M.$$

Conversely, any subfactor planar algebra P *with modulus* δ *arises from an extremal subfactor of index* δ^2 *in this fashion.*

We note that one consequence of this theorem is that for $x \in P_k = N' \cap M_{k-1}$, the pictorial trace $\tau(x)$ agrees with the canonical trace $tr_{M_{k-1}}(x)$. As an illustration of how this theorem is used, we prove the following proposition. We need the 2-rotation tangle R_2^2 which is depicted in Fig. 7. We also use the notation $P_{1,k}$ for $ran(Z_{(E')_k^k}) \subset P_k$.

Proposition 3.6. *Let* $N \subset M$ *be an extremal, finite index subfactor with planar algebra* P. *Set* $T = Z_{R_2^2}: P_2 \to P_2$. *Then* T *is an involutive, anti-algebra map that commutes with* $*$ *and maps* $P_1 \subset P_2$ *onto* $P_{1,2}$.

Proof. That T is involutive follows from $R_2^2 \circ R_2^2 = I_2^2$—where we omit the subscript to $\circ$ since R_2^2 has only one internal box. That T is an anti-algebra map follows from 'compatibility with renumbering internal boxes' and the fact that $R_2^2 \circ M_{2,2}^2 = (M^{op})_{2,2}^2 \circ_{(D_1,D_2)} (R_2^2, R_2^2)$ where $(M^{op})_{2,2}^2$ is the tangle obtained from $M_{2,2}^2$ by renumbering D_1 and D_2 as D_2 and D_1, respectively. That T commutes with $*$ follows from $(R_2^2)^* = R_2^2$.

To see that P_1 maps into $P_{1,2}$, note that P_1 is identified with $ran(Z_{I_1^2}) \subset P_2$ while $P_{1,2}$ is $ran(Z_{(E')_2^2}) \subset P_2$ so the desired result follows by observing that $(E')_2^2 \circ R_2^2 \circ I_1^2$ differs from $(E')_2^2$ only by a closed loop that contributes a multiplicative factor of δ.

To see that this is onto $P_{1,2}$ note that $(E')^2_2 = R^2_2 \circ I^2_1 \circ E^1_2$. $\square$

Before ending this section, we wish to recall the so-called *Fourier transform* maps [BscJns] between $N' \cap M_1$ and $M' \cap M_2$. Consider the map ψ: $N' \cap M_1 \to M' \cap M_2$ defined by

$$\psi(a) = \delta^3 E_{M' \cap M_2}(ae_2e_1)$$

for $a \in N' \cap M_1$. This map is described pictorially in Fig. 10.

One checks easily using Theorem 3.5 that the map φ: $M' \cap M_2 \to N' \cap M_1$ defined by

$$\varphi(b) = \delta^3 E_{N' \cap M_1}(be_1e_2)$$

for $b \in M' \cap M_2$ is a two-sided inverse of ψ.

We will use, in particular, that an arbitrary element of $M' \cap M_2$ is representable as $\psi(a)$ for $a \in N' \cap M_1$.

4. Duality and C^*-WHA structure of relative commutants

Throughout this section, we fix a finite index inclusion $N \subset M$ of II_1-factors of index δ^2 and of depth 2 and let $P = P^{N \subset M}$ be the associated planar algebra. Our interest will be confined to the algebras (and inclusions):

$$\begin{array}{ccccccc} & & & P_3 & & & \\ & & \nearrow & & \nwarrow & & \\ & P_2 & & & & P_{1,3}. & \\ \nearrow & & \nwarrow & & \nearrow & & \\ P_1 & & & P_{1,2} & & & \end{array}$$

It is a consequence of Theorem 3.5 that each of these is a finite-dimensional C^*-algebra and is related to the basic construction tower of the inclusion

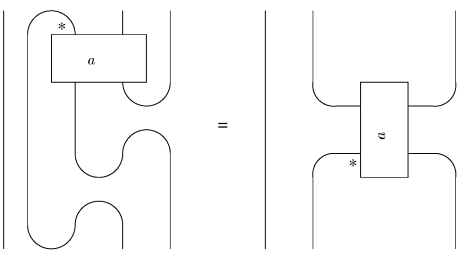


Fig. 10. Definition of $\psi(a)$.

$N \subset M$ by the equalities:

$$P_1 = N' \cap M,$$

$$P_2 = N' \cap M_1,$$

$$P_3 = N' \cap M_2,$$

$$P_{1,2} = M' \cap M_1$$

and

$$P_{1,3} = M' \cap M_2.$$

We will denote P_2 by A and $P_{1,3}$ by B and show that with their usual C^*-algebra structures, there exist mutually dual C^*-WHA structures on A and B.

We begin by introducing certain elements of $P_{1,2}$ and P_1 that play an important role throughout the proof. The C^*-algebra $P_{1,2}$ is endowed with two faithful positive trace functionals—one being the so-called canonical trace or the trace in the left regular representation which we will denote by τ_c and the other being the pictorial trace which we denote as usual by τ.

Let z_R be the Radon-Nikodym derivative of τ_c with respect to τ, by which we mean the unique element of $P_{1,2}$ for which $\tau_c(x) = \tau(z_R x)$ for all $x \in P_{1,2}$. It is easy to see that z_R is a well-defined, central, positive, invertible element of $P_{1,2}$. By w_R we denote the unique positive square root of z_R which also is a central, positive, invertible element of $P_{1,2}$. Let w_L be $Z_{R_2}(w_R)$ which is in P_1. We will use w to denote $w_L w_R^{-1}$. Note that as $w_R \in P_{1,2}$ and $w_L \in P_1$, they commute with each other. We will use, without mention, that w_R, w_L and w commute with all elements of P_1 and $P_{1,2}$.

Proposition 4.7. *There exist unique maps Δ: $A \to A \otimes A$, ε: $A \to \mathbb{C}$ and S: $A \to A$ such that the equations of Figs. 11 and 12 hold in the planar algebra P for all $a, x, y \in A$.*

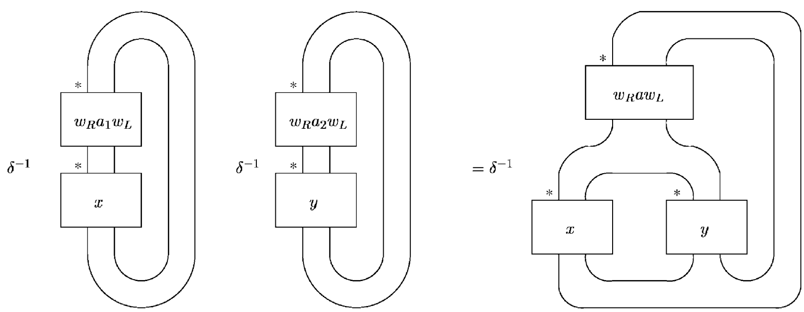


Fig. 11. The definition of Δ.

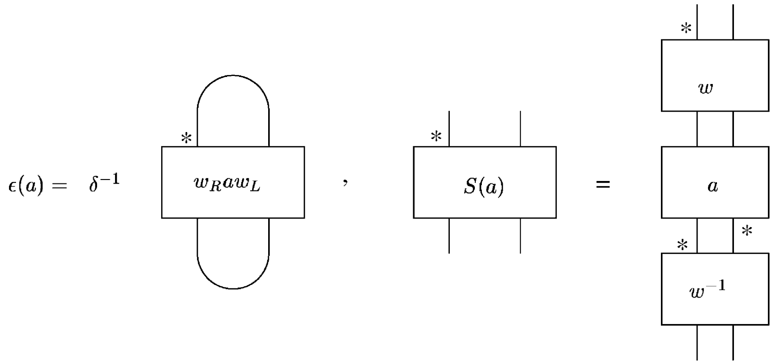


Fig. 12. Definitions for ε and S.

Before discussing the proof, we make some remarks regarding notations and the interpretation of such labelled tangle equations. Firstly, we will henceforth suppress drawing the external box of a tangle. It will be understood—for a k tangle with $k>0$—that the $*$ is at the top left corner. Secondly, a labelled tangle equation as in Fig. 11 is to be interpreted as:

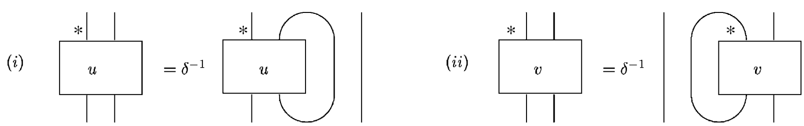
$$\delta^{-1}Z_W(w_{\mathrm{R}}aw_{\mathrm{L}}\otimes x\otimes y)=\delta^{-1}Z_{(tr_2^{0+})}(w_{\mathrm{R}}a_1w_{\mathrm{L}}x)\cdot\delta^{-1}Z_{(tr_2^{0+})}(w_{\mathrm{R}}a_2w_{\mathrm{L}}y)$$

in the planar algebra P where W is the tangle appearing in the LHS of Fig. 11 without labels. Finally, labelled 0-tangles in a connected planar algebra will be identified with complex numbers.

Proof of Proposition 4.7. It is clear that the equations in Fig. 12 define maps $\varepsilon : A\to\mathbb{C}$ and $S : A\to A$. That the equation in Fig. 11 defines a map $\Delta : A\to A\otimes A$ is a consequence of the facts that τ is a non-degenerate functional and $w_{\mathrm{L}}, w_{\mathrm{R}}$ are invertible elements of A. $\square$

We will now verify, by a series of propositions, that A, with the Δ, ε and S defined in Proposition 4.7, is a C^*-WHA. Most of the proofs will be pictorial. We start by observing that for elements $v\in P_{1,2}=ran(Z_{(E')_2^2})$, it follows from Theorem 3.5 that $v=\delta^{-1}Z_{(E')_2^2}(v)$.

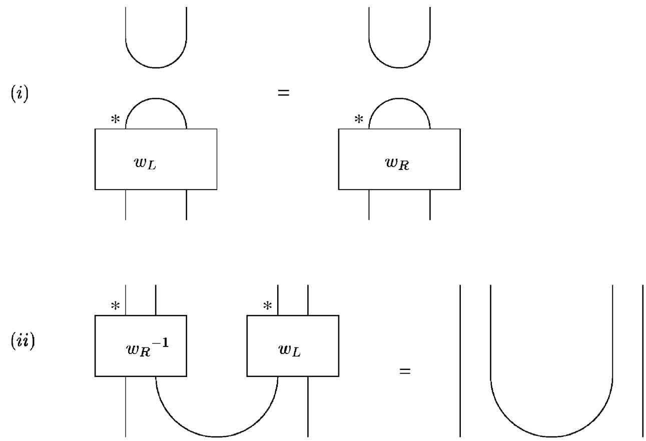
Hence a 2-box labelled by v actually behaves like a bead on its second strand. Similarly $u\in P_1$ is supported on the first strand. Thus, for $u\in P_1$ and $v\in P_{1,2}$, we have the pictorial equations:



It will be convenient to list a few consequences of this. We will isolate them in the next lemma and leave the proof to the reader.

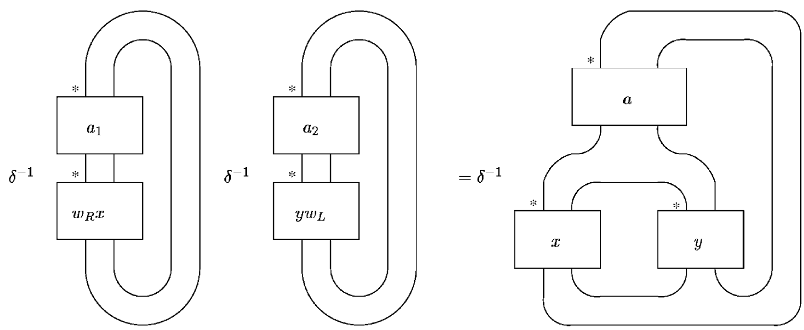
Lemma 4.8.

(a) *The following pictorial equalities hold*:



(b) $w_R a_1 \otimes a_2 w_L = a_1 w_R \otimes w_L a_2$ *for* $a \in A$.

(c) *The definition of* Δ *can be restated as*



(d) $\Delta(av) = \Delta(a)(1 \otimes v)$ *for* $a \in A$ *and* $v \in P_{1,2}$

Note that Lemma 4.8(a)(i) says that $e_1 w_L = e_1 w_R$. Also, for the sake of brevity we mentioned only one of the four allowable moves of its kind in 4.8(a)(ii). One more such move would be with the labels w_R and w_L^{-1} in the boxes in place of w_R^{-1} and w_L, respectively, and two more for the case where the boxes are connected by strings coming out from the top. Finally, the interpretation of Lemma 4.8(a)(ii) is that in any tangle picture where the LHS occurs, it may be replaced by the RHS.

Proposition 4.9. *The maps Δ and ε defined in Proposition* 4.7 *endow A with a coalgebra structure.*

Before proving this proposition, we introduce a pairing between A and B that will be shown to give the duality. The idea to use a non-degenerate pairing in this setting is due to Szymański [Szy]. Close relatives of the pairing defined here also appear in [KdsNks,NksVnr].

For $a \in A$ and $b \in B$, define

$$\langle a \,|\, b \rangle = \delta^4 \tau(e_2 e_1 b w_{\mathrm{R}} a w_{\mathrm{L}}). \tag{4.2}$$

As each element $b \in B$ is representable as $\psi(T(x))$ for some $x \in A$ the expression in Eq. (4.2) is equivalent to Fig. 13. It is easy to see that this is a non-degenerate pairing using that τ is a non-degenerate functional on A and the invertibility of w_{L} and w_{R}.

Proof of Proposition 4.9. It suffices to see that with respect to the pairing defined, Δ and ε are the duals of the multiplication and unit of B. Writing two typical elements of B as $\psi(Tx)$ and $\psi(Ty)$ the picture on the right in Fig. 11 represents $\langle a \,|\, \psi(T(x))\psi(T(y)) \rangle$ while the picture on the left represents $\langle \Delta(a) \,|\, \psi(T(x)) \otimes \psi(T(y)) \rangle$. This proves the statement about Δ and a similar verification with Fig. 12 establishes the statement about ε. $\square$

Remark 4.10. It is possible—and very instructive—to give a proof of Proposition 4.9 pictorially without using the duality results for Δ and ε.

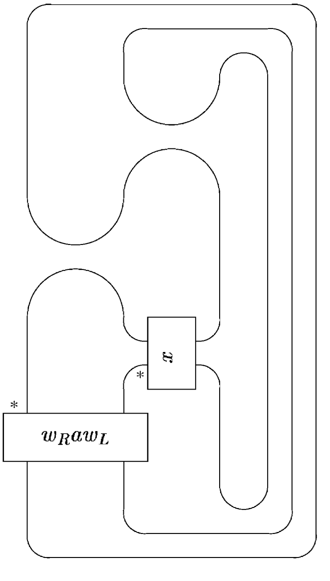
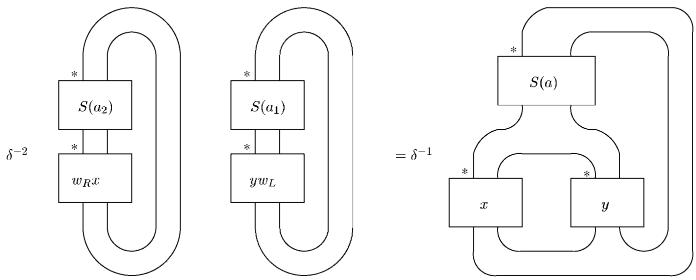


Fig. 13. Pairing between a and $b = \psi(x)$.

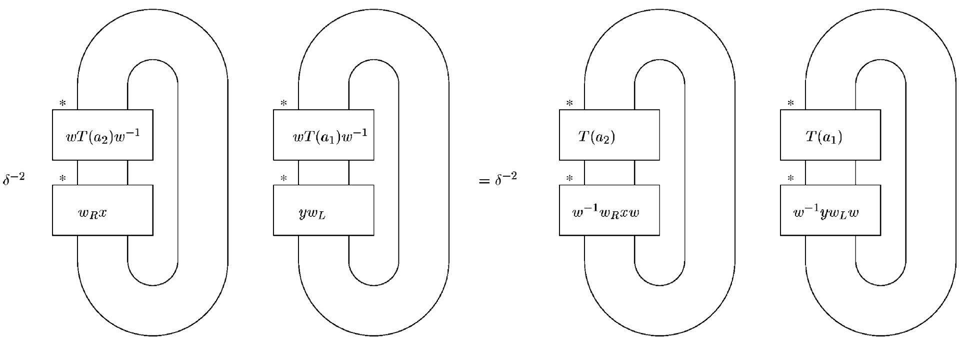
Proposition 4.11. *The map S: $A \to A$ defined in Proposition* 4.7 *is an anti-algebra and anti-coalgebra map that agrees with $T = Z_{R_2^2}$ on P_1 and $P_{1,2}$.*

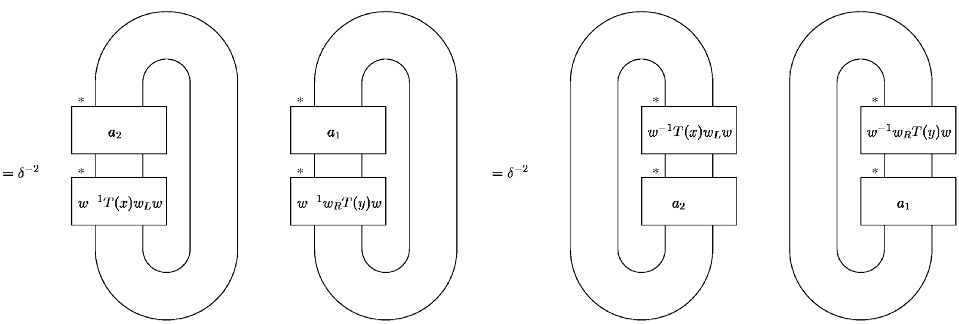
Proof. A very easy verification using Fig. 12, that we omit, shows that S is an anti-algebra map. To see that it is an anti-coalgebra map we need to see that $\Delta \circ S = (S \otimes S) \circ \Delta^{op}$ or equivalently that $(S(a))_1 \otimes (S(a))_2 = S(a_2) \otimes S(a_1)$.

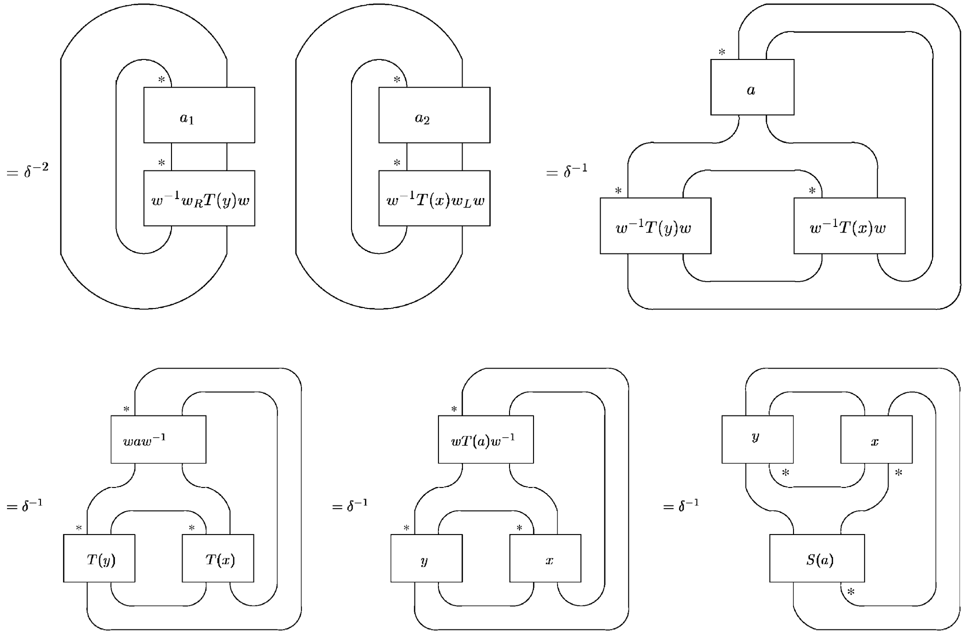
By Lemma 4.8(c) we are reduced to proving the following pictorial identity for all $a, x, y \in A$.



Note that by definition, $S(a) = wT(a)w^{-1}$ for $a \in A$ and that $T(w) = w^{-1}$. Therefore, the LHS equals







which equals the RHS. The third and eighth equality follow since $T = Z_{R_2}$, the fourth and fifth by isotopy invariance and the sixth by the definition of Δ. The others hold by moving 2-boxes labelled by w and w^{-1} appropriately.

That S agrees with T restricted to P_1 and $P_{1,2}$ follows from Proposition 3.6 and commutativity of w with elements of P_1 and $P_{1,2}$. $\square$

We now proceed towards proving that Δ is multiplicative. This is proved by bootstrapping by first showing that $\Delta(1)^2 = \Delta(1)$, using that to prove an 'exchange relation' for Δ and using that to prove multiplicativity. We begin by identifying $\Delta(1)$.

Recall that a symmetric separability element of an algebra A is an element $f^{(1)} \otimes f^{(2)} \in A \otimes A$ such that $xf^{(1)} \otimes f^{(2)} = f^{(1)} \otimes f^{(2)}x$ for all $x \in A$, $f^{(1)}f^{(2)} = 1$, and $f^{(1)} \otimes f^{(2)} = f^{(2)} \otimes f^{(1)}$. It is easy to see that $f^{(1)} \otimes f^{(2)}$ is an idempotent if considered as an element of $A \otimes A^{op}$. We clarify that we are using 'Sweedler like' notation with the summation sign suppressed. In particular, $f^{(1)} \otimes f^{(2)}$ stands for an arbitrary element of $A \otimes A$ and not just a decomposable element.

Finite-dimensional C^*-algebras have a unique symmetric separability element which is also the 'quasi-basis' with respect to the canonical trace τ_c. This just means that for all elements $x \in A$ one has

$$f^{(1)}\tau_c(f^{(2)}x) = x = \tau_c(xf^{(1)})f^{(2)}.$$

For the C^*-algebra $P_{1,2}$, since $z_R = w_R^2$ is the Radon–Nikodym derivative of τ_c with respect to τ it follows that the quasi-basis of τ is $w_R f^{(1)} \otimes w_R f^{(2)}$ or equivalently that

$$w_{\mathrm{R}} f^{(1)} \tau(w_{\mathrm{R}} f^{(2)} x) = x = \tau(x w_{\mathrm{R}} f^{(1)}) w_{\mathrm{R}} f^{(2)},$$

for all $x \in P_{1,2}$.

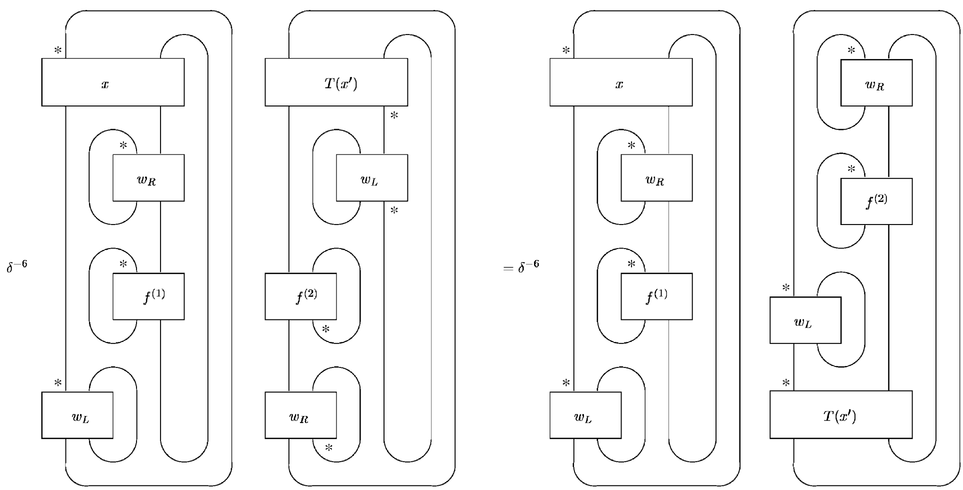
The next lemma is very similar to [KdsNks, Eqn(33)].

Proposition 4.12. *$\Delta(1) = f^{(1)} \otimes S(f^{(2)})$ where $f^{(1)} \otimes f^{(2)}$ is the symmetric separability element of $P_{1.2}$.*

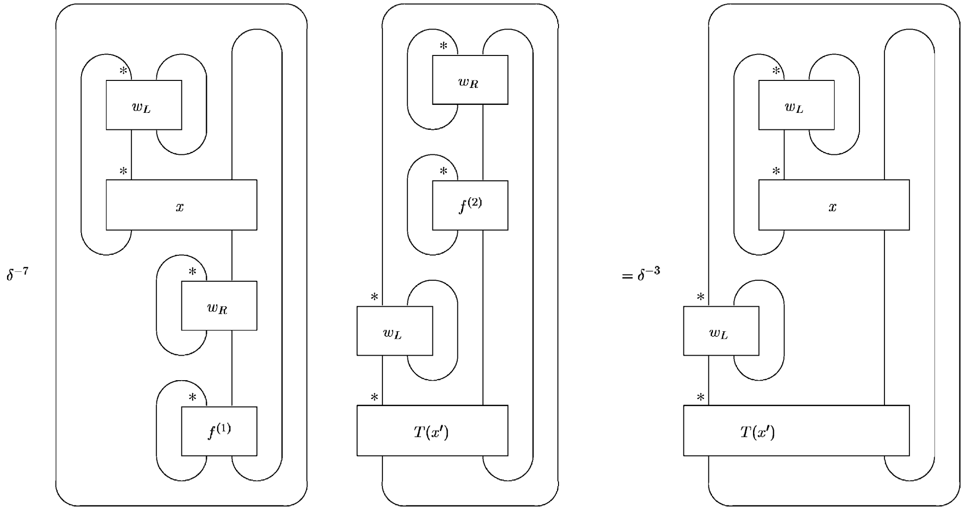
Proof. By Proposition 4.9, since Δ is dual to the multiplication in B it suffices to see that

$$\langle f^{(1)} \mid b \rangle \langle S(f^{(2)}) \mid b' \rangle = \langle 1 \mid bb' \rangle \forall b, b' \in B.$$

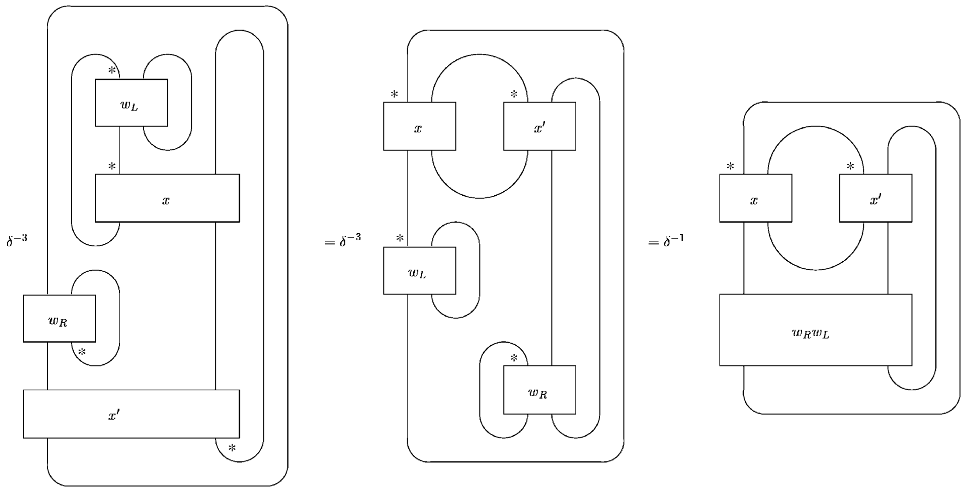
Writing b and $b' \in P_{1,2}$ as $\psi(T(x))$ and $\psi(T(x'))$, respectively, with $x, x' \in P_2$ and using Eq. (4.2) and Fig. 13 the LHS translates to the following picture:



One has to be careful keeping track of the multiplicative constant of δ in writing out these pictorial equations. In the above picture the presence of the constant is due to the pictorial equations (i) and (ii) prior to Lemma 4.8. The second equality is obtained by noting that S agrees with T on P_1 and using sphericity. Again, since a loop contributes a multiplicative factor of δ, we have,



Here, equality follows from the quasi-basis statement. In the last figure, definition of T is applied to change it to:



The last picture correctly interpreted is just the RHS. $\square$

Corollary 4.13. *$\Delta(1)$ is in $P_{1,2} \otimes P_1$ and is an idempotent.*

Proof. Since S maps $P_{1,2}$ to P_1, it follows from Proposition 4.12 that $\Delta(1)$ is in $P_{1,2} \otimes P_1$. To see that it is an idempotent, from the definition of separability element, we have that for all $a \in P_{1,2}$,

$$(a \otimes 1)(f^{(1)} \otimes f^{(2)}) = (f^{(1)} \otimes f^{(2)})(1 \otimes a)$$

$$\Rightarrow f^{(1)}f^{(1')} \otimes f^{(2')}f^{(2)} = f^{(1')} \otimes f^{(2')}f^{(1)}f^{(2)}$$

$$= f^{(1')} \otimes f^{(2')},$$

Now,

$$(\Delta(1))^2 = (f^{(1)} \otimes S(f^{(2)}))(f^{(1')} \otimes S(f^{(2')}))$$

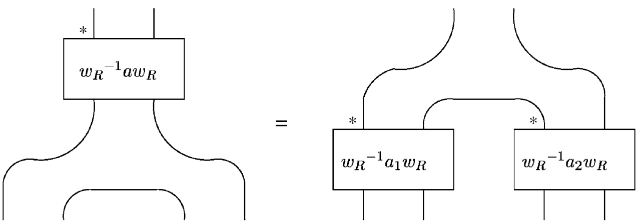
$$= f^{(1)}f^{(1)'} \otimes S(f^{(2')}f^{(2)})$$

$$= f^{(1)} \otimes S(f^{(2)})$$

$$= \Delta(1),$$

where the fifth equality holds since S is an anti-algebra map. $\square$

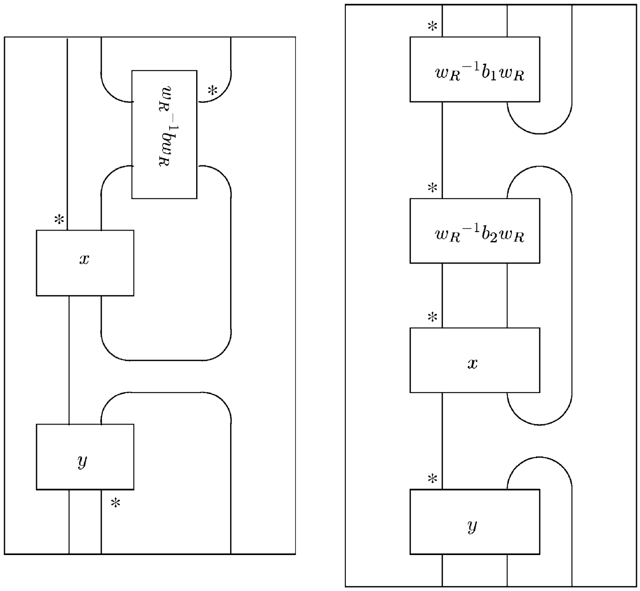
We will find it convenient to say that the exchange relation holds for $a \in A$ if the following pictorial equation holds:



in the planar algebra P.

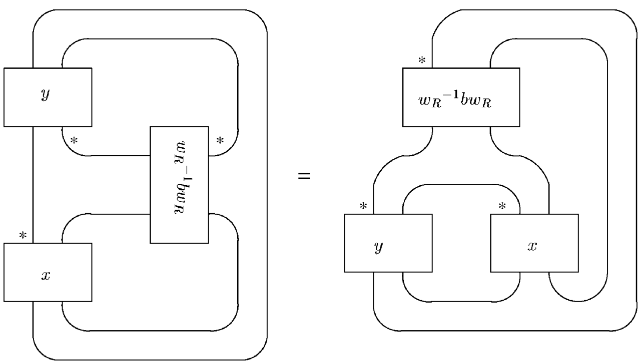
Lemma 4.14. *For $b \in A$, if $\Delta(1)\Delta(b) = \Delta(b)$, then the exchange relation holds for b.*

Proof. The exchange relation asserts an equality between two elements of P_3. Since τ is non-degenerate and the depth two condition implies that an arbitrary element of P_3 may be represented as $xe_2T(y)$ for $x, y \in P_2$, it is enough to check whether LHS and RHS of the following diagram are the same after applying tr_3^{0+}.

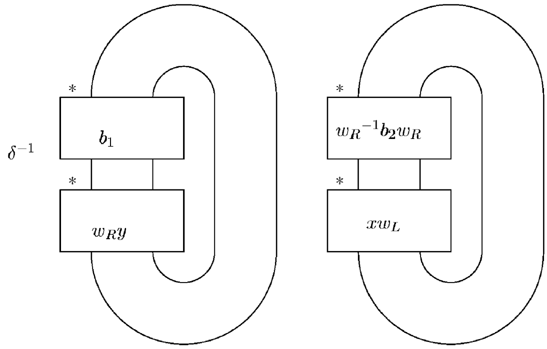


LHS RHS

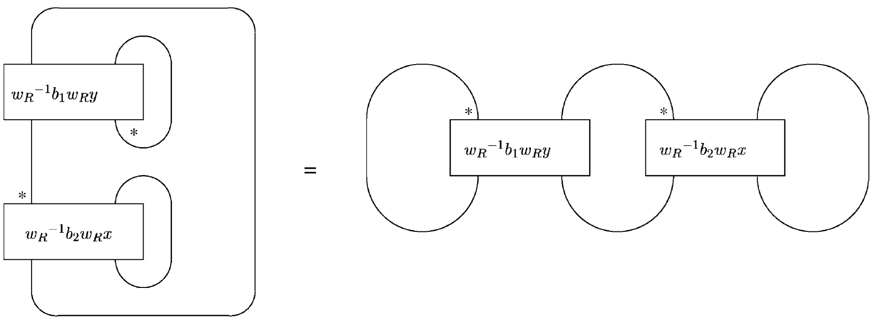
After closing up and moving the box with label y on top the LHS becomes

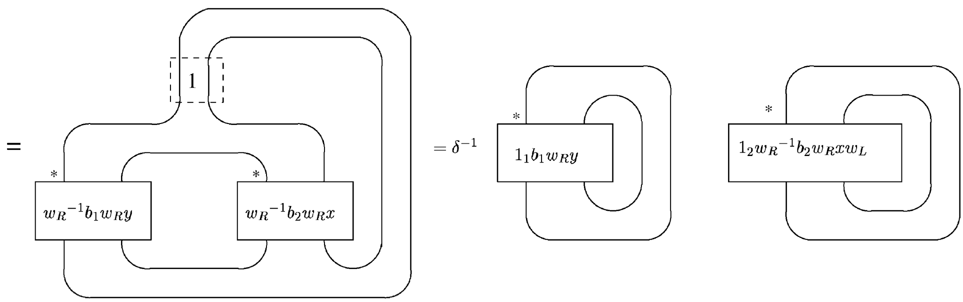


by sphericity. Now, by definition of Δ this equals



After performing similar moves to the right-hand side we get





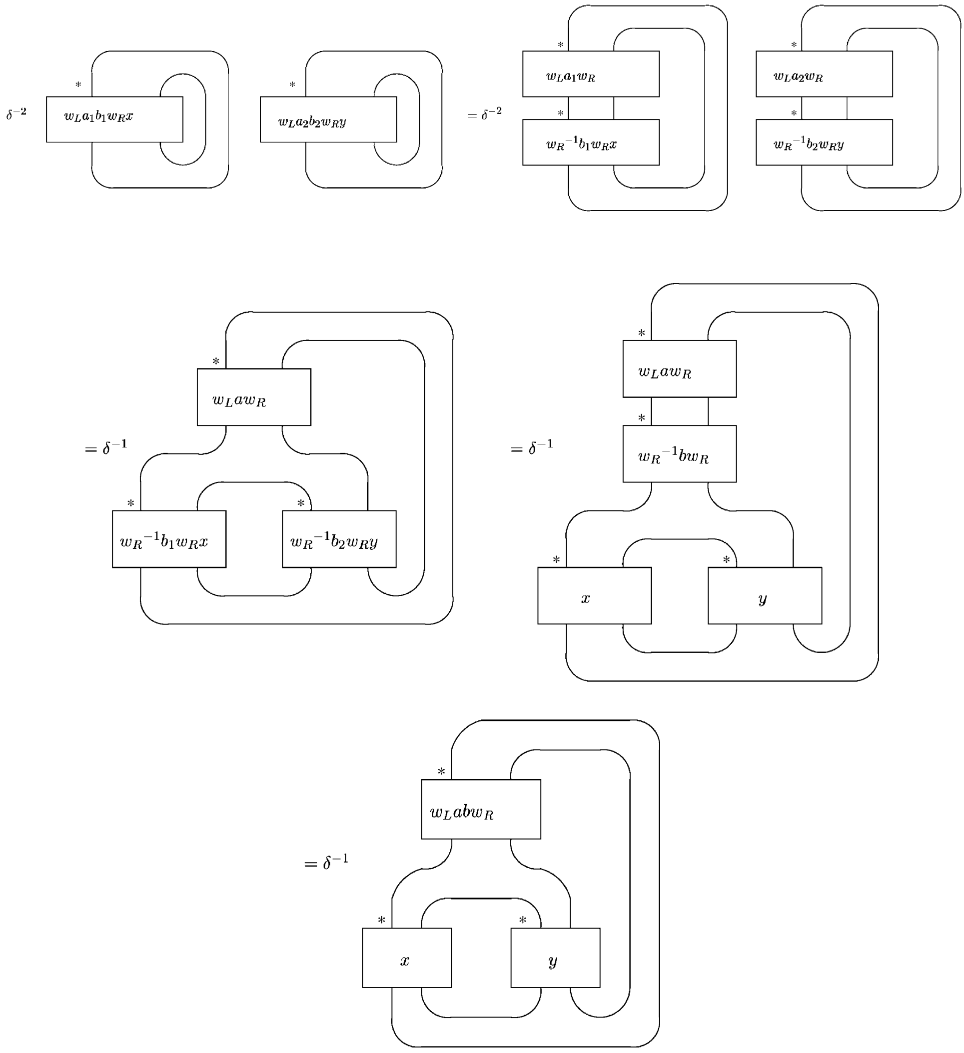
where we have used sphericity in the first and the second equalities. It now follows from Corollary 4.13 that the two sides are equal once we have

$$1_1 b_1 \otimes 1_2 b_2 = b_1 \otimes b_2$$

which is exactly what has been assumed. $\square$

Lemma 4.15. *If the exchange relation holds for $b \in A$, then $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a \in A$.*

Proof. Consider the following pictorial equations:



The second equality is by definition of Δ while the third uses the exchange relation for b. Comparing the first and last terms and again using the definition of Δ completes the proof. $\square$

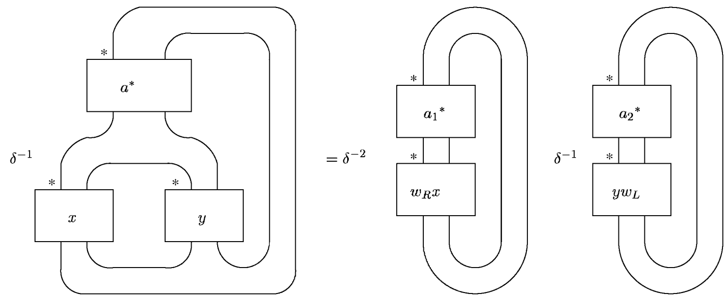
Proposition 4.16. (i) *The exchange relation holds for all* $b \in A$.
(ii) $\Delta(ab) = \Delta(a)\Delta(b)$ *for all* $a, b \in A$.

Proof. (ii) follows from (i) and Lemma 4.15. To prove (i), note that Corollary 4.13 and Lemma 4.14 imply that the exchange relation holds for $1 \in A$. By Lemma 4.15,

$\Delta(b) = \Delta(b)\Delta(1)$ for all $b \in A$. Applying $\tau \circ (S \otimes S)$, using Proposition 4.11 and replacing $S(b)$ by b gives $\Delta(b) = \Delta(1)\Delta(b)$ for all $b \in A$. Again appeal to Lemma 4.14 to complete the proof. $\square$

Proposition 4.17. *Δ: $A \to A \otimes A$ is a $*$-homomorphism.*

Proof. Given Proposition 4.16(ii) it remains only to see that $\Delta(a^*) = \Delta(a)^*$ or equivalently that $(a^*)_1 \otimes (a^*)_2 = (a_1)^* \otimes (a_2)^*$. This is equivalent to the following pictorial equality.

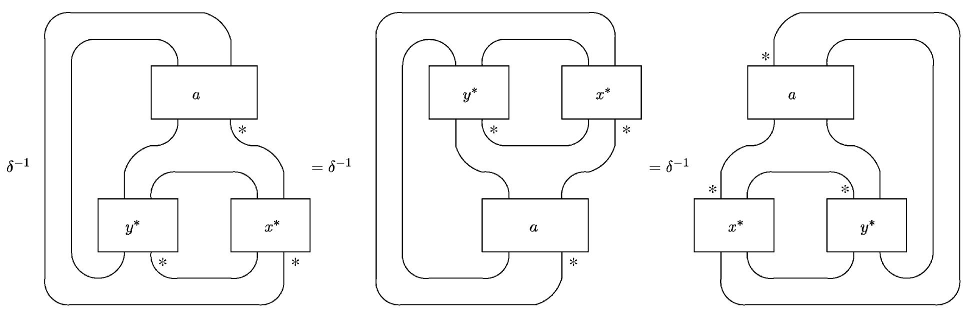


Note that it is enough to show equality after taking $*$ of both the sides. Recall that

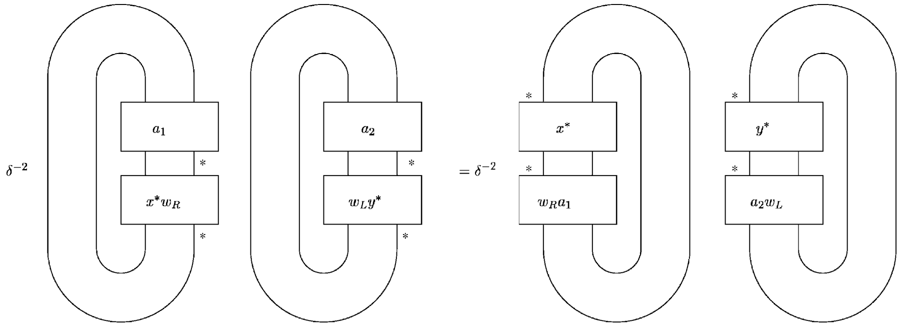
$$Z_T(x_1 \otimes x_2 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes x_2^* \otimes \cdots \otimes x_b^*),$$

where T^* is the tangle obtained from T by reflecting and putting $*$ on all the k-boxes in the reflected $2k$-positions.

Taking $*$ of the LHS we have



while the $*$ of the RHS gives



By Lemma 4.8(b), $w_{\mathrm{R}}a_1 \otimes a_2 w_{\mathrm{L}} = a_1 w_{\mathrm{R}} \otimes w_{\mathrm{L}} a_2$ and so the two sides are equal using the definition of Δ. $\square$

Corollary 4.18. $(\Delta \otimes id)(\Delta(1)) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))$.

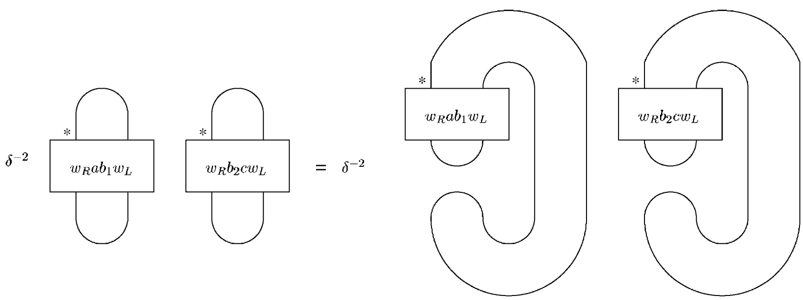
Proof.

$$
\begin{aligned}
RHS &= 1_1 \otimes 1_2 1_{1'} \otimes 1_{2'} \\
&= (\Delta(1)(1 \otimes 1_{1'})) \otimes 1_{2'} \\
&= \Delta(1_{1'}) \otimes 1_{2'} \\
&= (\Delta \otimes id)(\Delta(1)) \\
&= LHS,
\end{aligned}
$$

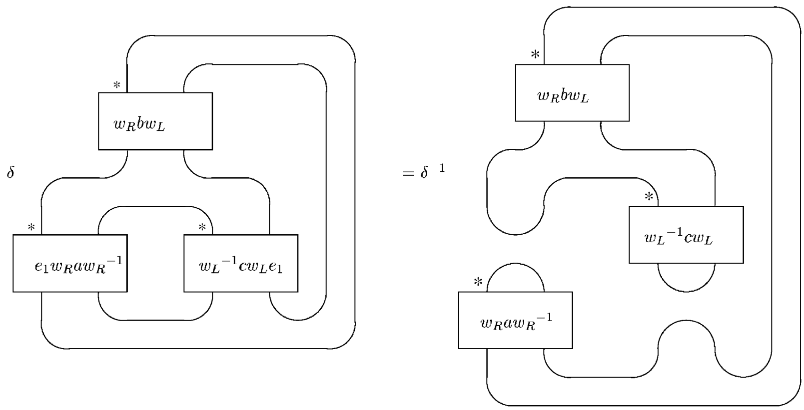
where the third equality is a consequence of Lemmas 4.12 and 4.8(d). $\square$

Proposition 4.19. *The counital map* ε: $A \to \mathscr{C}$ satisfies $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2 c)$.

Proof. $\varepsilon(ab_1)\varepsilon(b_2 c) =$



Now, we write $w_R ab_1 w_L$ as $(w_R a w_R^{-1})(w_R b_1 w_L)$, $w_R b_2 c w_L$ as $(w_R b_2 w_L)(w_L^{-1} c w_L)$ and move the appropriate boxes down below to get that the RHS of the above pictorial equality is



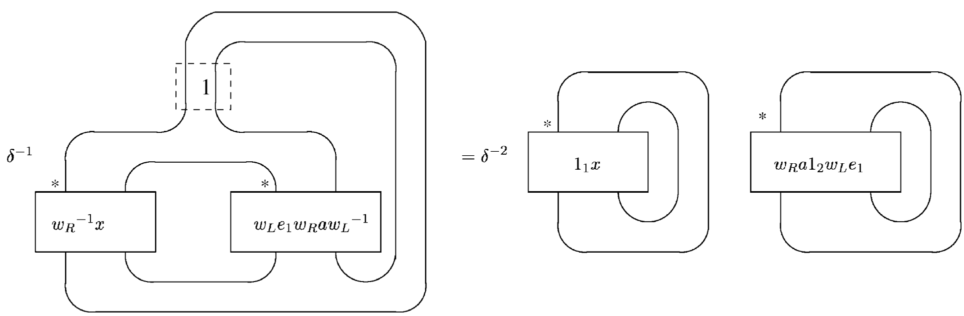
which is precisely $\varepsilon(abc)$ after moving the strings isotopically. □

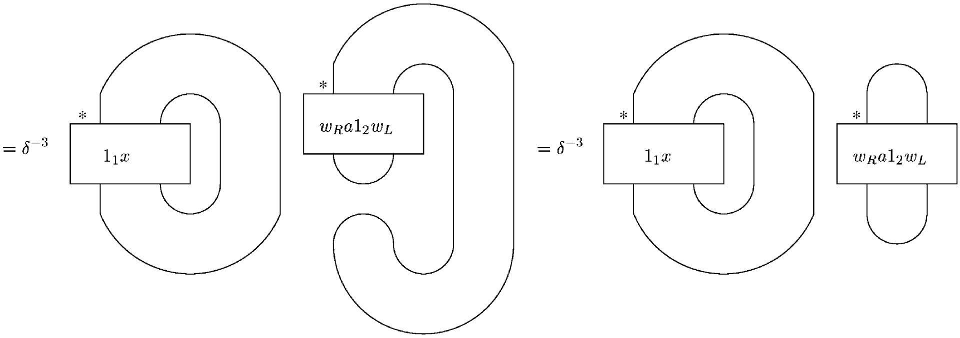
Proposition 4.20. *For all $a \in A$, $S(a_1)a_2 = 1_1 \varepsilon(a 1_2)$.*

Proof. It suffices to show that the traces of both the sides after multiplying with an arbitrary $x \in A$ are the same. i.e., we need to see that

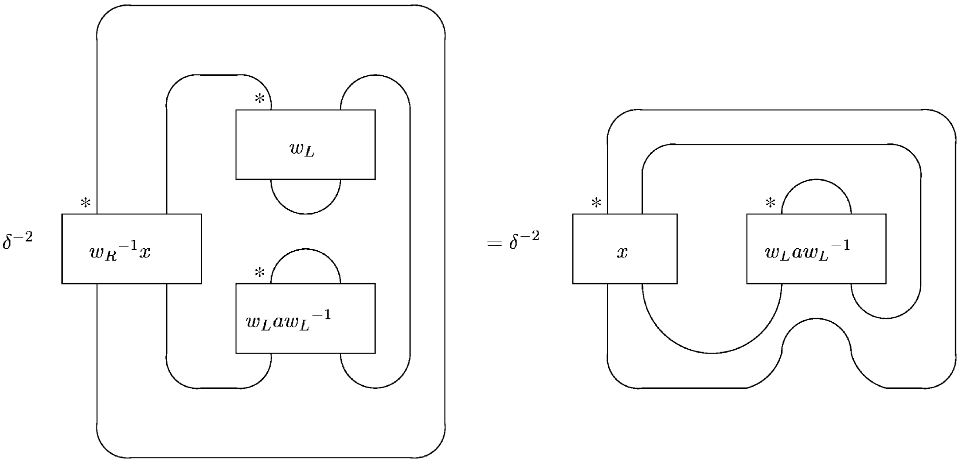
$$\tau(S(a_1)a_2 x) = \tau(1_1 x)\varepsilon(a 1_2).$$

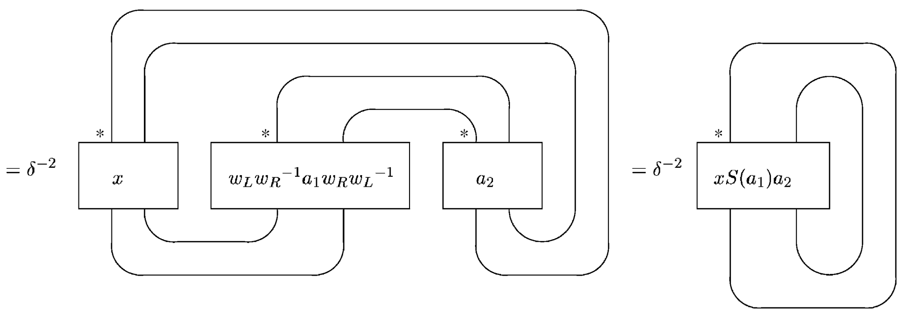
Starting with the picture below, using first the definition of Δ, then that of e_1 and finally isotopy we get the following string of equalities:





which is precisely $\tau(1_1x)\varepsilon(a1_2)$. On the other hand, noting that $e_1w_{\mathrm{R}} = e_1w_{\mathrm{L}}$ by Lemma 4.8(a)(i), the first picture on the first line simplifies to





The first equality follows from an analogue of Lemma 4.8(a)(ii), the second from exchange relation and the third from the fact that $T(wa_1w^{-1}) = S(a_1)$ to yield $\tau(S(a_1)a_2x)$. $\square$

Theorem 4.21. *Let $N \subset M$ be a depth two II_1 subfactor of finite index. Then $A = N' \cap M_1$ is a C^*-WHA.*

Proof. With the definions of Δ, ε and S as in Proposition 4.7 and its natural C^*-algebra structure, axioms (1)–(5) of Definition 2.1 for A are verified in Proposition 4.17, Corollary 4.18 and Propositions 4.19, 4.11 and 4.20, respectively. $\square$

We will next consider the relation between the $*$-structures of A and B and the WHA structure.

Proposition 4.22. *For $a \in A$ and $b \in B$, $\langle a \mid b^* \rangle = \overline{\langle S(a^*) \mid b \rangle}$.*

Proof. Let an arbitrary element $b \in B$ be of the form $\psi(x)$ for some $x \in A$. Then the prescription of the * structure of a planar algebra as in Eq. (3.1) implies that b^* is precisely $\psi(T(x^*))$.

To avoid drawing any more pictures, we denote the labelled 0-tangle in Fig. 13 by $Z_Q(x \otimes a)$. It represents the scalar $\delta^2 \tau(T(x) w_{\mathrm{R}} a w_{\mathrm{L}})$.

It follows that

$$\begin{aligned}\langle a \mid b^* \rangle &= Z_Q(T(x^*) \otimes a)\\ &= \delta^2 \tau(x^* w_{\mathrm{R}} a w_{\mathrm{L}})\\ &= \delta^2 \tau(w_{\mathrm{R}} T(a) w_{\mathrm{L}} T(x^*)),\end{aligned}$$

where the last equality holds since $\tau = \tau \circ T$. On the other hand,

$$\begin{aligned}\langle S(a^*) \mid b \rangle &= Z_Q(x \otimes S(a^*))\\ &= \delta^2 \tau(T(x) w_{\mathrm{R}} S(a^*) w_{\mathrm{L}})\end{aligned}$$

and so

$$\overline{\langle S(a^*) \mid b \rangle} = \delta^2 \tau(w_{\mathrm{L}} S^{-1}(a) w_{\mathrm{R}} T(x^*)).$$

We want to mention here that it is a consequence of Definition 2.1 that $(S \circ *)^2 = id$ in A. The last equality is because of this and the fact that the $*$ of a labelled 0-tangle is the same as complex conjugation of the scalar it represents.

Now, equality of the two sides follows from the definition of S. $\square$

We now gather everything together in our main result.

Theorem 4.23. *Let $N \subset M$ be a depth two II_1 subfactor of finite index. Then $A = N' \cap M_1$ and $B = M' \cap M_2$ admit structures of C^*-WHAs that are dual in the sense of Definition 2.4.*

Proof. We have seen that A has a C^*-WHA structure and that with respect to the duality defined between A and B, the relation $\langle a|b^*\rangle = \overline{\langle S(a^*)\,|\,b\rangle}$ holds. By the remarks after Definition 2.4, the proof is complete. $\square$

It should be mentioned at this point that in the case of an irreducible subfactor one recovers Szymański's result through these computations.

5. The action on M

We will now verify that—in the hyperfinite case—the C^*-WHA $P_2 = N' \cap M_1$ acts on the factor M with invariants N. We should remark that while it follows from [KdsNks,NksVnr] that there is an action of P_2 on M with invariants N even in the case that N and M are not hyperfinite, our pictorial method of proof is limited to the hyperfinite case. The action here is built out of actions on $P_{1,n}$ and Popa's theorem—which requires hyperfiniteness—is used to construct an action on M.

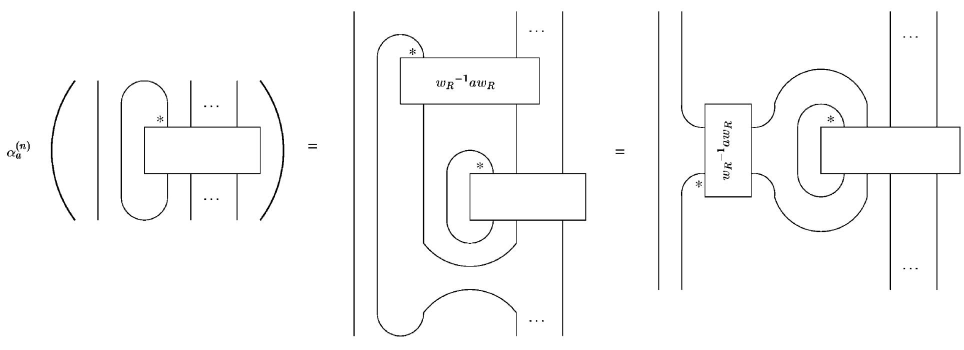
We begin by recalling the definition of an action of a C^*-WHA on a factor due to [NllSzlWsb].

Definition 5.24. By an action of a C^*-WHA A on a II_1 factor M we mean a map α: $A \to End_{\mathbb{C}}(M)$ (where $\alpha(a)$ is denoted by α_a) satisfying the following conditions for all $a, b \in A$; $x, y \in M$:

(i) $\alpha_1 = id_M$,
(ii) $\alpha_{ab} = \alpha_a \circ \alpha_b$,
(iii) $\alpha_a(xy) = \alpha_{a_1}(x)\alpha_{a_2}(y)$,
(iv) $\alpha_a(x)^* = \alpha_{S(a)^*}(x^*)$,
(v) $\alpha_a(1_M) = \alpha_{\varepsilon_L(a)}(1_M)$.

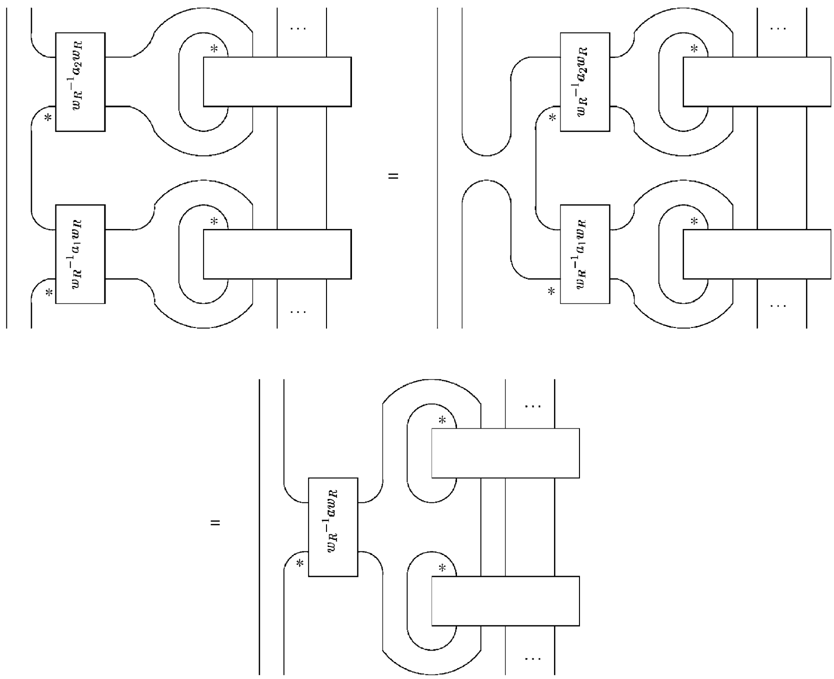
The invariant subalgebra for this action is defined to be the set $\{x \in M \text{ such that } \alpha_a(x) = \alpha_{\varepsilon_L(a)}(x) \forall a \in A\}$.

In order to define an action of P_2 on M, first define for $a \in P_2$, a map $\alpha_a^{(n)}$: $P_{1,n} \to P_{1,n}$ by the following picture:



It is easy to check that the $\alpha_a^{(n)}$'s so defined are compatible with the inclusions of $P_{1,n}$ into $P_{1,n+1}$ and thus patch up to define maps α_a from $\bigcup_n(M' \cap M_n)$ to itself. The extended map α turns out to be an *anti-action* of P_2 on $\bigcup_n(M' \cap M_n)$, which just means that if we define β_a: $(M' \cap M_k)^{op} \to (M' \cap M_k)^{op}$ by $\beta_a(x^{op}) = \alpha_a(x)^{op}$ then β is an action.

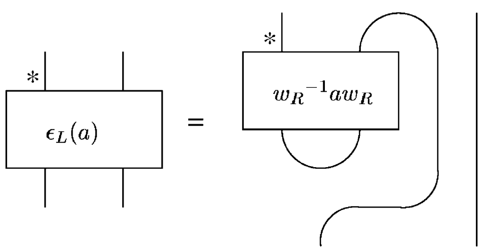
Properties (i) and (ii) of Definition 5.24 are obvious. In place of (iii), we have to show that $\alpha_a(xy) = \alpha_{a_2}(x)\alpha_{a_1}(y)$. This is a direct application of the exchange relation is illustrated in the next picture.



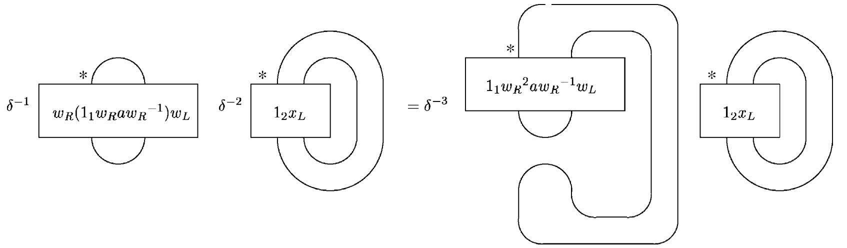
Property (iv) is a consequence of the definition of the adjoint tangle and the relation between the maps T and S.

For proving (v) and that the invariants for this anti-action are exactly $M_1{}' \cap M_n$ we need to take a closer look at the left counital map of the C^*-WHA P_2.

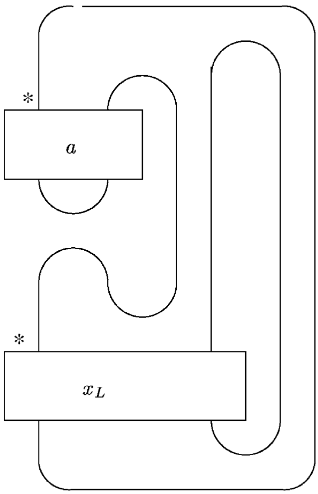
Lemma 5.25. *The following pictorial relation holds* $\forall a \in P_2$:



Proof. It is enough to show equality after replacing a by $w_R a w_R^{-1}$ and pairing against an arbitrary $x_L \in P_1$. After doing so, the LHS is $\tau(\varepsilon_L(w_R a w_R^{-1})x_L) = \varepsilon(1_1 w_R a w_R^{-1})\tau(1_2 x_L) =$



$= \delta^2 \tau(1_1 w_R^2 a w_R^{-1} w_L e_1)\tau(1_2 x_L) = \delta^2 \tau(1_1 w_R^2 a e_1)\tau(1_2 x_L)$ whereas the RHS is



which is just $\delta^2 \tau(x_L a e_1)$. As this should hold $\forall a \in A$, we demand that $e_1 x_L = e_1 w_R^2 1_1 \tau(1_2 x_L)$, which holds because of the quasi-basis statement and the fact that $e_1 x_L = e_1 S(x_L)$. $\square$

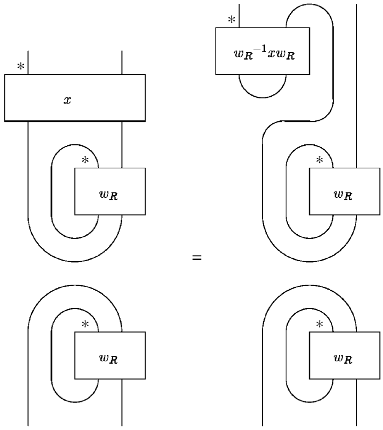
Now (v) follows, as the two sides of the picture in Lemma 5.25 remain equal after ‘capping off’ the strings at the bottom.

Lemma 5.25 implies that elements of $M_1{}' \cap M_n$ are invariants in the sense of Definition 5.24.

Conversely, if x is an invariant, then it must follow that $\alpha_h(x) = x$ where h is the normalised, two-sided integral which we identify in the next Lemma.

Lemma 5.26. *The normalised, two-sided integral of the C^*-WHA P_2 is given by $h = \frac{1}{\rho} w_R e_1 w_R$, where ρ is the dimension of $P_{1,2}$.*

Proof. First we show that it is a left integral, i.e., that $xh = \varepsilon_L(x)h \forall x \in A$. The equality in the following picture is a consequence of isotopy of strings:

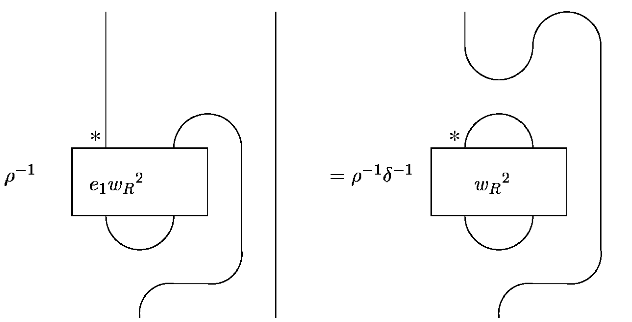


Now, h is a left integral implies that $S(h)$ is a right integral. But

$$\begin{aligned} S(h) &= \rho^{-1}S(w_R e_1 w_R) \\ &= \rho^{-1} w T(w_R e_1 w_R) w^{-1} \\ &= \rho^{-1} w w_L e_1 w_L w^{-1} \\ &= \rho^{-1} w_R e_1 w_R \\ &= h, \end{aligned}$$

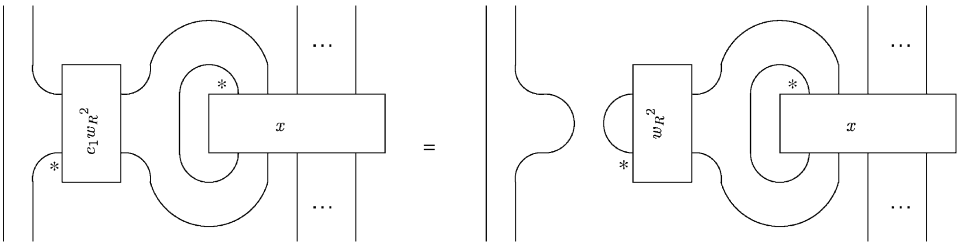
where we used the fact that $w_L e_1 = w_R e_1$ in the fourth and the fifth equalities. So, h is a two-sided integral.

To show that h is normalised, i.e., $\varepsilon_L(h) = 1$ note that $w_R^{-1}(w_R e_1 w_R)w_R = e_1 w_R^2$ and hence $\varepsilon_L(h) =$



which is just $\rho^{-1}\tau(w_R^2) = \rho^{-1}\tau_c(1) = 1$, since $\tau_c(1) = dim(P_{1,2}) = \rho$. $\square$

Let x be an invariant now. So, $\alpha_h(x) = x$, which—by the following picture—shows that x is actually in $M_1{}' \cap M_n$.



Now, extremality implies that the traces on $N' \cap M_n$ coming from N' and M_n are equal. Also, using Jones' Theorem the pictorial definition of action gets translated to $\alpha_a(x) = \delta^2 E_{M'}(w_{\mathrm{R}}^{-1} a w_{\mathrm{R}} x e_1)$ where $E_{M'}$ is the trace-preserving conditional expectation of N' onto M', and therefore α_a is continuous for the weak topology on N'. This should be compared with the action in Proposition 4.1 of [KdsNks] which was the motivation for its definition.

Now $\bigcup_n (M' \cap M_n)$ is weakly dense in $M' \cap M_\infty$. Hence α_a extends to a unique anti-action on $M' \cap M_\infty$ and has invariants $M_1{}' \cap M_\infty$. In the hyperfinite case, Popa's Theorem gives the anti-isomorphism between the pairs $M_1{}' \cap M_\infty \subset M' \cap M_\infty$ and $N \subset M$ which translates the anti-action of P_2 into an action on M with invariants N.

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